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# HIGH ENERGY RESOLVENT ESTIMATES FOR ACOUSTIC PROPAGATORS IN A STRATIFIED MEDIA

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## §1 Introduction.

Let  $n \geq 2$  and  $x = (y, z) \in \mathbf{R}^{n-1} \times \mathbf{R}$ . In this report we study the following operator :

$$(1.1) \quad L_0 = -a_0(z)^2 \Delta,$$

where

$$a_0(z) = \begin{cases} c_+ & (z \geq h) \\ c_h & (0 < z < h) \\ c_- & (z \leq 0), \end{cases}$$

and  $c_{\pm}, c_h$  and  $h$  are positive constants and

$$\Delta = \sum_{j=1}^{n-1} \frac{\partial^2}{\partial y_j^2} + \frac{\partial^2}{\partial z^2}.$$

We consider only the case  $c_h < \min(c_+, c_-)$  because we can find the guided waves (cf. Wilcox [9] or Weder [6]). It seems that there are no works dealing with high energy resolvent estimates for acoustic propagators in stratified media. Here we shall prove high energy resolvent estimates for the case  $c_h < c_+ = c_-$ .

Kikuchi-Tamura [3] and Kadowaki [2] have proved low energy resolvent estimates for the case  $c_h < \min(c_+, c_-)$  and  $c_+ \neq c_-$  and the case  $c_h < c_+ = c_-$  respectively. Both works were used Mourre's commutator method (cf. Mourre[4]). But the conjugate operator in Kadowaki [2] is different from Kikuchi-Tamura [3]. Kikuchi-Tamura [3] took the generator of dailation in  $\mathbf{R}^3$  as the conjugate operator. They dealt with only media of  $\mathbf{R}^3$  but their result can be extended for media of  $\mathbf{R}^n (n \geq 3)$  (cf. Kadowaki [2]). Kadowaki [2] has constructed the conjugate operator by using the generator of dailation in  $\mathbf{R}^n$  and  $\mathbf{R}^{n-1} (n \geq 3)$  together with the generalized Fourier transform of a related operator (cf. Weder [6]). The generator of dailation in  $\mathbf{R}^{n-1}$  has been used to estimate the guided wave (see §2). In this report, we also use Mourre's method. Our conjugate operator is similar to Kadowaki [2] ( see §2 ).

Let  $\mathcal{H}_0 = L^2(\mathbf{R}^n; a_0^{-2}(z)dx)$  be Hilbert space with inner products

$$\langle u, v \rangle_0 = \int_{\mathbf{R}^n} u(x) \overline{v(x)} a_0^{-2}(z) dx.$$

In particular  $L^2(\mathbf{R}_x^n)$  is the usual  $L^2$  space defined on  $\mathbf{R}_x^n$  with inner products

$$\langle u, v \rangle_{L^2(\mathbf{R}_x^n)} = \int_{\mathbf{R}_x^n} u(x) \overline{v(x)} dx$$

and the corresponding norms  $|\cdot|_{L^2(\mathbf{R}_x^n)}$ .

$L_0$  admits a unique self-adjoint realizations in  $\mathcal{H}_0$ . Then  $L_0$  is a non-negative operator (zero is not an eigenvalue) and the  $D(L_0)$  is given by  $H^2(\mathbf{R}_x^n)$ ,  $H^s(\mathbf{R}_x^n)$  being Sobolev space of order  $s$  over  $\mathbf{R}_x^n$ . We also denoted by  $R(z; L_0)$  the resolvent  $(L_0 - z)^{-1}$  of  $L$  for  $\text{Im} z \neq 0$ .

$A$  is considered as an operator from  $L^2(\mathbf{R}_x^n)$  into itself, then its norm is denoted by the notation  $\|A\|$ .

We remark that Weder [6] has showed the absence of eigenvalues and the limiting absorption principle for  $L_0$ . Our result is :

**Theorem 1.1.** *Let  $\alpha > 1/2$ . Assume that  $c_h < c_+ = c_-$ . Then, we have*

$$\|X_\alpha R(\lambda \pm i\kappa; L_0) X_\alpha\| = O(\lambda^{-\frac{1}{2}}) \quad (\lambda \rightarrow \infty),$$

uniformly in  $\kappa > 0$ , where  $X_\alpha = (1 + |x|^2)^{-\alpha/2}$ .

We define the self-adjoint operator  $L_0(\lambda)$  on  $L^2(\mathbf{R}_x^n)$ ,

$$\begin{cases} L_0(\lambda) = -\Delta - \lambda(a_0^{-2}(z) - c_+^{-2}) \\ D(L_0(\lambda)) = H^2(\mathbf{R}_x^n). \end{cases}$$

This operator has been introduced by Weder [7]. Theorem 1.1 is obtained as an immediate consequence of the following proposition.

**Proposition 1.2.** *Assume that  $c_h < c_+ = c_-$ . Then we have*

$$\|X_\alpha G_\kappa(0; \lambda) X_\alpha\| = O(\lambda^{-\frac{1}{2}}) \quad (\lambda \rightarrow \infty),$$

uniformly in  $\kappa > 0$ , where

$$G_\kappa(0; \lambda) = (L_0(\lambda) - \lambda c_+^{-2} - i\kappa a_0^{-2}(z))^{-1}$$

for  $\kappa > 0$

In §§2,3 we shall give the proof of above proposition.

We give a comment for the assumption of Theorem 1.1. This follows from our method. Applying Mourre's method to the original operator,  $L_0$ , we do not get the Mourre's estimates on the neighborhood of thresholds of  $L_0$  (cf. Wilcox [9] or Weder [6]). The conjugate operator for  $L_0$  is constructed by using generator of dilation in  $\mathbf{R}^n$  and exterior domains of ball in  $\mathbf{R}^{n-1}$  together with the generalized Fourier transform for  $L_0$  (cf. Kadowaki [1]). While, since  $L_0(\lambda)$  does not have thresholds on  $[0, \infty)$  (see Weder [7]), we can obtain Mourre's estimates. But, to prove only Lemma 3.6 in §3, we need the assumption  $c_h < c_+ = c_-$ . In brief we deal with only  $c_h < c_+ \leq c_-$ .

As an application of our theorem, we can consider scattering problem for wave equations with dissipative terms in stratified media. This is due to Mochizuki [4]. He has proved existence of scattering states for wave equations with dissipative terms in the case  $c_h = c_+ = c_- = 1$ . His idea is due to Kato's smooth perturbation theory together with low and high energy resolvent estimates for Laplacian in  $\mathbf{R}^n (n \neq 2)$ . To consider scattering problem for stratified media, we need low energy estimates which is required in Mochizuki [4]. Kikuchi-Tamura [3] and Kadowaki [2] have proved low energy estimates in perturbed stratified media. But the 3-dimensional case in Kadowaki [3] and Kikuchi-Tamura [2] do not satisfy Mochizuki's condition (for detail see Mochizuki [4]). For Kikuchi-Tamura's result, we can remake it to satisfy Mochizuki's condition (see Kadowaki [3]). We will give low energy estimates for stratified media of  $\mathbf{R}^n (n \geq 2)$  elsewhere and consider scattering problem.

## §2 Conjugate operator and Mourre's estimates.

In this section we construct the conjugate operators and show Mourre's estimates (2.1). First we define conjugate operator,  $D(\lambda)$ , as follows :

$$D(\lambda) = F_0(\lambda)^*(-D_n)F_0(\lambda) + F_1(\lambda)^*(-D_{n-1})F_1(\lambda) \\ + \sum_{j=1}^{Q(\lambda)} G_j(\lambda)^*(-D_{n-1})G_j(\lambda),$$

where  $k = (\bar{k}, k_0) \in \mathbf{R}^{n-1} \times \mathbf{R}$ ,  $F_0(\lambda)$ ,  $F_1(\lambda)$  and  $G_j(\lambda)$  are partially isometric operators for  $L_0(\lambda)$  (see Appendix ) and

$$D_n = \frac{1}{2i}(k \cdot \nabla_k + \nabla_k \cdot k), \quad D_{n-1} = \frac{1}{2i}(\bar{k} \cdot \nabla_{\bar{k}} + \nabla_{\bar{k}} \cdot \bar{k}).$$

We consider the commutator  $i[L_0(\lambda), D(\lambda)]$  as a form on  $H^2(\mathbf{R}_x^n) \cap D(D(\lambda))$  as follows :

$$\begin{aligned} & \langle i[L_0(\lambda), D(\lambda)]u, u \rangle_{L^2(\mathbf{R}_x^n)} \\ &= i(\langle D(\lambda)u, L_0(\lambda)u \rangle_{L^2(\mathbf{R}_x^n)} - \langle L_0(\lambda)u, D(\lambda)u \rangle_{L^2(\mathbf{R}_x^n)}) \end{aligned}$$

for  $u \in H^2(\mathbf{R}^n) \cap D(D(\lambda))$ . Then Lemma A of the Appendix implies that

$$\begin{aligned} & \langle i[L_0(\lambda), D(\lambda)]u, u \rangle_{L^2(\mathbf{R}_x^n)} \\ &= i\{ \langle |k|^2 F_0(\lambda)u, D_n F_0(\lambda)u \rangle_{L^2(\mathbf{R}_k^n)} - \langle D_n F_0(\lambda)u, |k|^2 F_0(\lambda)u \rangle_{L^2(\mathbf{R}_k^n)} \\ &+ \langle |\bar{k}|^2 F_1(\lambda)u, D_{n-1} F_1(\lambda)u \rangle_{L^2(\Omega_0)} - \langle D_{n-1} F_1(\lambda)u, |\bar{k}|^2 F_1(\lambda)u \rangle_{L^2(\Omega_0)} \\ &+ \sum_{j=1}^{Q(\lambda)} (\langle |\bar{k}|^2 G_j(\lambda)u, D_{n-1} G_j(\lambda)u \rangle_{L^2(\mathbf{R}_k^{n-1})} \\ &- \langle D_{n-1} G_j(\lambda)u, |\bar{k}|^2 G_j(\lambda)u \rangle_{L^2(\mathbf{R}_k^{n-1})}) \}. \end{aligned}$$

Thus we have by integral by parts

$$\begin{aligned} & \langle i[L_0(\lambda), D(\lambda)]u, u \rangle_{L^2(\mathbf{R}_x^n)} \\ &= \langle 2(F_0(\lambda)^*|k|^2 F_0(\lambda) + F_1(\lambda)^*|\bar{k}|^2 F_1(\lambda) + \sum_{j=1}^{Q(\lambda)} G_j(\lambda)^*|\bar{k}|^2 G_j(\lambda))u, u \rangle_{L^2(\mathbf{R}_x^n)} \end{aligned}$$

for  $u \in H^2(\mathbf{R}_x^n) \cap D(D(\lambda))$ . Thus the form  $i[L_0(\lambda), D(\lambda)]$  can be extended to a bounded operator from  $H^1(\mathbf{R}_x^n)$  to  $H^{-1}(\mathbf{R}_x^n)$  which is denoted by  $i[L_0(\lambda), D(\lambda)]^0$ . Let  $\lambda > 1$ , take  $f_\lambda(r) \in C_0^\infty(\mathbf{R})$ ,  $0 \leq f_\lambda \leq 1$  such that  $f_\lambda$  has support in  $((c_+^{-2} - c_-^{-2}/2)\lambda, 2c_+^{-2}\lambda)$  and  $f_\lambda = 1$  on  $[(c_+^{-2} - c_-^{-2}/4)\lambda, 3c_+^{-2}\lambda/2]$ . Noting that

$$\begin{aligned} & f_\lambda(L_0(\lambda))i[L_0(\lambda), D(\lambda)]^0 f_\lambda(L_0(\lambda)) \\ &= 2(F_0(\lambda)^*|k|^2 f_\lambda(|k|^2 + q_-(\lambda))^2 F_0(\lambda) + F_1(\lambda)^*|\bar{k}|^2 f_\lambda(|\bar{k}|^2 - k_0^2 + q_-(\lambda))^2 F_1(\lambda) \\ &+ \sum_{j=1}^{Q(\lambda)} G_j(\lambda)^*|\bar{k}|^2 f_\lambda(|\bar{k}|^2 - \omega_j^2(\lambda))^2 G_j(\lambda)). \end{aligned}$$

Then there exists a positive constant  $C$  which is independent of  $\lambda$  such that

$$(2.1) \quad f_\lambda(L_0(\lambda))i[L_0(\lambda), D(\lambda)]^0 f_\lambda(L_0(\lambda)) \geq C\lambda f_\lambda(L_0(\lambda))^2$$

in the form sense.

### §3 Proof of Proposition 1.2.

Proposition 1.2 follows from lemmas in this section. But we omit the proof of lemmas and give only a comment of the proof.

We can prove the following lemmas in the same way as in the proof of Lemma 2.5 of Weder [7].

**Lemma 3.1.** *Let  $f \in C_0^\infty(\mathbf{R})$ . Then*

- (i)  $f(L_0(\lambda))$  sends  $D(D(\lambda))$  into  $D(D(\lambda))$ .
- (ii)  $[f(L_0(\lambda)), D(\lambda)]$  defined as operator on  $D(D(\lambda))$  is extended to a bounded operator on  $L^2(\mathbf{R}_x^n)$  which is denoted by  $[f(L_0(\lambda)), D(\lambda)]^0$ .

It follows from (2.1) that  $M_0(\lambda)$  is non-negative and hence we define an operator,  $G_\kappa(\epsilon; \lambda)$ , on  $L^2(\mathbf{R}_x^n)$  by

$$(3.1) \quad G_\kappa(\epsilon; \lambda) = (L_0(\lambda) - \lambda c_+^{-2} - i\kappa a_0^{-2}(z) - i\epsilon M_0(\lambda))^{-1}$$

for  $\kappa > 0$  and  $\epsilon > 0$ . Using (2.1), we can prove the following lemma (for detail, see that of Lemma 5.3 of Kikuchi-Tamura [3]).

**Lemma 3.2.** *For  $\epsilon > 0$ , as  $\lambda \rightarrow \infty$ , one has*

$$\|G_\kappa(\epsilon; \lambda)\| = \epsilon^{-1}O(\lambda^{-1}), \quad (\lambda \rightarrow \infty)$$

uniformly in  $\kappa > 0$ .

We write

$$F_\kappa(\epsilon; \lambda) = \lambda^{\frac{1}{2}} Z_\alpha(\epsilon, \lambda) G_\kappa(\lambda^{-\frac{1}{2}} \epsilon; \lambda) Z_\alpha(\epsilon, \lambda),$$

where  $Z_\alpha(\epsilon, \lambda) = (\lambda^{\frac{1}{2}} + |D(\lambda)|)^{-\alpha} (\lambda^{\frac{1}{2}} + \epsilon |D(\lambda)|)^{\alpha-1}$ .

This is due to Yafaev [8]. But we do not use the scaling argument for  $\lambda$  (cf. (3.1)).

Let  $g_\lambda(p) = 1 - f_\lambda(p)$ . We write in brief  $f_\lambda$  and  $g_\lambda$  for  $f_\lambda(L_0(\lambda))$  and  $g_\lambda(L_0(\lambda))$  respectively.

Noting that

$$G_\kappa(\epsilon; \lambda)D(D(\lambda)) \subset D(D(\lambda)) \cap H^2(\mathbf{R}^n)$$

(cf. Kadowaki [2]), we decompose  $(d/d\epsilon)F_\kappa(\epsilon; \lambda)$  as a form on  $L^2(\mathbf{R}_x^n)$

$$(3.2) \quad (d/d\epsilon)F_\kappa(\epsilon; \lambda) = \sum_{j=1}^8 Y_\kappa^j(\epsilon; \lambda),$$

where

$$\begin{aligned} Y_\kappa^1 &= iZ_\alpha(\epsilon, \lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)g_\lambda[L_0(\lambda), D(\lambda)]^0 f_\lambda G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)Z_\alpha(\epsilon, \lambda), \\ Y_\kappa^2 &= iZ_\alpha(\epsilon, \lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)g_\lambda[L_0(\lambda), D(\lambda)]^0 g_\lambda G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)Z_\alpha(\epsilon, \lambda), \\ Y_\kappa^3 &= iZ_\alpha(\epsilon, \lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)f_\lambda[L_0(\lambda), D(\lambda)]^0 g_\lambda G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)Z_\alpha(\epsilon, \lambda), \\ Y_\kappa^4 &= -iZ_\alpha(\epsilon, \lambda)\{D(\lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda) + G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)D(\lambda)\}Z_\alpha(\epsilon, \lambda) \\ Y_\kappa^5 &= \kappa Z_\alpha(\epsilon, \lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)[a_0(z)^{-2}, D(\lambda)]G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)Z_\alpha(\epsilon, \lambda), \\ Y_\kappa^6 &= \epsilon Z_\alpha(\epsilon, \lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)[M_0(\lambda), D(\lambda)]G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)Z_\alpha(\epsilon, \lambda), \\ Y_\kappa^7 &= \lambda^{-\frac{1}{2}}\left\{\frac{d}{d\epsilon}Z_\alpha(\epsilon, \lambda)\right\}G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)Z_\alpha(\epsilon, \lambda), \\ Y_\kappa^8 &= \lambda^{-\frac{1}{2}}Z_\alpha(\epsilon, \lambda)G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)\frac{d}{d\epsilon}Z_\alpha(\epsilon, \lambda). \end{aligned}$$

We need the following lemmas (Lemma 3.3~Lemma 3.5) to estimate each term of the right side of (3.2).

Note that there is  $c_0, c_0 > 0$  such that  $(L_0(\lambda) + c_0\lambda)^{-1}$  exists.

**Lemma 3.3.** *As  $\lambda \rightarrow \infty$ , one has :*

- (i)  $\|g_\lambda G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)\| = O(\lambda^{-1})$ ,
  - (ii)  $\|(L_0(\lambda) + c_0\lambda)^{1/2}f_\lambda G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)Z_\alpha(\epsilon, \lambda)\| = \epsilon^{-1/2}\|F_\kappa\|^{1/2}O(1)$ ,
  - (iii)  $\|(L_0(\lambda) + c_0\lambda)^{1/2}g_\lambda G_\kappa(\lambda^{-\frac{1}{2}}\epsilon; \lambda)Z_\alpha(\epsilon, \lambda)\| = O(\lambda^{-1})$ ,
  - (iv)  $\|F_\kappa(\epsilon; \lambda)\| = \epsilon^{-1}O(\lambda^{-1})$ ,
- uniformly in  $\kappa > 0$ .

For a proof of Lemma 3.5 (i), see that of Lemma 5.4 of Kikuchi-Tamura [3]. Also, for a proof of (ii) and (iii), see that of Lemma 5.5 of Kikuchi-Tamura [3]. (ii) and (iii) imply (iv).

**Lemma 3.4.** *Assume that  $c_h < c_+ = c_-$ . Then  $[a_0^{-2}(z), D(\lambda)]$  defined as a form on  $D(D(\lambda))$  is extended to a bounded operator from  $H^1(\mathbf{R}_x^n)$  to  $H^{-1}(\mathbf{R}_x^n)$  which is denoted by  $[a_0^{-2}(z), D(\lambda)]^0$ . Moreover we have*

$$\begin{aligned} & i[a_0^{-2}(z), D(\lambda)]^0 \\ &= (c_h^{-2} - 1)((n-1)\chi_{0 < z < h}(z) - (\partial_{k_0}F_0(\lambda)\chi_{0 < z < h}(z))^*k_0F_0(\lambda) \\ & \quad - (k_0F_0(\lambda))^*\partial_{k_0}F_0(\lambda)\chi_{0 < z < h}(z) + F_0(\lambda)^*F_0(\lambda)) \end{aligned}$$

and

$$\|(L_0(\lambda) + c_0\lambda)^{-1/2}i[a_0^{-2}(z), D(\lambda)]^0(L_0(\lambda) + c_0\lambda)^{-1/2}\| = O(\lambda^{-\frac{1}{2}}) \quad (\lambda \rightarrow \infty).$$

*proof.* Noting that the representation of  $F_0(\lambda)$ , we show this lemma by straightforward calculation (cf. Kadowaki [2]).

Using Lemma 3.1 and the representation of  $i[L_0(\lambda), D(\lambda)]^0$  we show the following lemma.

**Lemma 3.5.** *As  $\lambda \rightarrow \infty$ , one has :*

$$\|[M_0(\lambda), D(\lambda)]^0\| = O(\lambda).$$

Using Lemma 3.2 ~ 3.5, we can evaluate the norm of  $Y_\kappa^j$ ,  $1 \leq j \leq 8$  (see Kikuchi-Tamura [3]). Thus we obtain the following differential inequality :

$$(3.3) \quad \|(d/d\epsilon)F_\kappa(\epsilon; \lambda)\| \leq C(\lambda^{-1}\epsilon^{\alpha-1} + \lambda^{-\frac{1}{2}}\epsilon^{\alpha-\frac{3}{2}}\|F_\kappa\|^{1/2} + \|F_\kappa\|)$$

It follows from Lemma 3.3(iv) and (3.3) that

$$(3.4) \quad \|(\lambda^{\frac{1}{2}} + |D(\lambda)|)^{-\alpha} G_\kappa(0; \lambda) (\lambda^{\frac{1}{2}} + |D(\lambda)|)^{-\alpha}\| = O(\lambda^{-\frac{1}{2}-\alpha}), \quad (\lambda \rightarrow \infty),$$

uniformly in  $\kappa > 0$ .

Noting Lemma 3.1 we rewrite  $D(\lambda)f_\lambda X_1$  as

$$(3.5) \quad \begin{aligned} & \frac{1}{i}(f_\lambda \nabla_y \cdot y X_1 + \frac{n-1}{2} f_\lambda X_1) \\ & - \frac{1}{i}(f_\lambda F_0(\lambda)^* k_0 \partial_{k_0} F_0(\lambda) X_1 + \frac{1}{2} f_\lambda F_0(\lambda)^* F_0(\lambda) X_1) \\ & + [D(\lambda), f_\lambda]^0 X_1. \end{aligned}$$

We can show that

$$(3.6) \quad \|[D(\lambda), f_\lambda]^0\| = O(1), \quad (\lambda \rightarrow \infty),$$

(for proof, see that of Lemma 5.6 of Kikuchi-Tamura [3]).

By straightforward calculation we can show next lemma.

**Lemma 3.6.** *As  $\lambda \rightarrow \infty$ , one has :*

$$\|f_\lambda F_0(\lambda)^* k_0 \partial_{k_0} F_0(\lambda) X_1\| = O(\lambda^{\frac{1}{2}})$$

It follows from (3.5), (3.6) and Lemma 3.8 that

$$\|D(\lambda)f_\lambda X_1\| = O(\lambda^{\frac{1}{2}}) \quad (\lambda \rightarrow \infty).$$

Thus we obtain by interpolation

$$(3.) \quad \|(\lambda^{\frac{1}{2}} + |D(\lambda)|)^\alpha f_\lambda X_\alpha\| = O(\lambda^{\frac{\alpha}{2}}) \quad (\lambda \rightarrow \infty).$$

Note that

$$\|g_\lambda G_\kappa(0; \lambda)\| = O(\lambda^{-1}) \quad (\lambda \rightarrow \infty).$$

(3.4) and (3.7) imply that

$$\|X_\alpha G_\kappa(0; \lambda) X_\alpha\| = O(\lambda^{-\frac{1}{2}}) \quad (\lambda \rightarrow \infty).$$

Now the proof of Proposition 1.2 is complete.

### Appendix.

In this Appendix we state the generalized Fourier transform of  $L_0(\lambda)$  established by Weder (cf. Weder [6]).

For  $\lambda \gg 1$  large enough, we consider the following operator :

$$\begin{cases} h(\lambda) = -\frac{d^2}{dz^2} - \lambda(a_0^{-2}(z) - c_+^{-2}), \\ D(h(\lambda)) = H^2(\mathbf{R}_z). \end{cases}$$

This is the self-adjoint operator in  $L^2(\mathbf{R}_z)$ .

$h(\lambda)$  has finite number  $Q(\lambda) \in \mathbf{N}$ , of eigenvalues,  $-\omega_j^2(\lambda)$ ,  $0 < \omega_j^2(\lambda) < q_h(\lambda) = \lambda(c_h^{-2} - c_+^{-2})$ ,  $1 \leq j \leq Q(\lambda)$ , of multiplicity one. There exist  $F_0(\lambda)$ ,  $F_1(\lambda)$  and  $G_j(\lambda)$  ( $j = 1, 2, 3 \dots Q(\lambda)$ ) which are partially isometric operators from  $L^2(\mathbf{R}_x^n)$  onto  $L^2(\mathbf{R}_k^n)$ ,  $L^2(\Omega_0)$  and  $L^2(\mathbf{R}_k^{n-1})$  respectively, where  $\Omega_0 = \{k \in \mathbf{R}^n; 0 < k_0 < \sqrt{q_-(\lambda)} = \sqrt{\lambda(c_+^{-2} - c_-^{-2})}\}$ . Defining the operator  $F(\lambda)$  as

$$F(\lambda)u = (F_0(\lambda)u, F_1(\lambda)u, G_1(\lambda)u, G_2(\lambda)u, G_3(\lambda)u \dots G_{Q(\lambda)}u(\lambda))$$

for  $u \in L^2(\mathbf{R}_x^n)$ , we have

**Lemma A.**  $F(\lambda)$  is unitary operator from  $L^2(\mathbf{R}_x^n)$  onto

$$\hat{\mathcal{H}} = L^2(\mathbf{R}_k^n) \bigoplus L^2(\Omega_0) \bigoplus_{j=1}^{Q(\lambda)} L^2(\mathbf{R}_k^{n-1})$$

and for every  $u \in D(L_0(\lambda)) = H^2(\mathbf{R}_x^n)$

$$\begin{aligned} F(\lambda)L_0(\lambda)u &= ((|k|^2 + q_-(\lambda))F_0(\lambda)u, (|\bar{k}|^2 - k_0^2 + q_-(\lambda))F_1(\lambda)u, \\ &(|\bar{k}|^2 - \omega_1^2(\lambda))G_1(\lambda)u, (|\bar{k}|^2 - \omega_2^2(\lambda))G_2(\lambda)u, \dots, \\ &(|\bar{k}|^2 - \omega_{Q(\lambda)}^2(\lambda))G_{Q(\lambda)}(\lambda)u). \end{aligned}$$

For the proof, see Weder [6].

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